Hysteresis as an Authentic Mathematics Application

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Integrating authentic applications in mathematics teaching is an important part of student learning because it supports classroom participation, engagement with assessments and greater retention, which leads to an overall increased interest in the subject (Campbell, Patterson, Bushch-Vishniac, & Kibler, 2008). To support this, a real-world practical application called hysteresis is suggested as a motivation for learning a variety of mathematics topics. Hysteresis is well-known in an engineering or physics discipline because of its connections to physical processes; however it is not normally discussed within a mathematics framework.

To begin, a brief overview of hysteresis is presented. This includes two physical examples that motivate hysteresis, and an explanation of how hysteresis is mathematically modelled. After this, some suggestions are noted of why hysteresis might be an ideal application to stimulate the teaching of fundamental math concepts like functions, geometry, graphs, and computational modelling.

To note, in the discussions that follow, a function f is understood to mean a rule that assigns to each element $x \in D$ exactly one element, called $f(x)$, that belongs in E, where D and E are sets of the real numbers, denoted $\mathbb{R}$ (Stewart, 2012, p.10). It will be explained later on that hysteresis inspires what is not a function, which is often less of a focus in the context of teaching functions. This leads to general pedagogical reflection about teaching functions through applications rather than by definition, and raises such followup questions as how to use manipulatives to aid in the teaching of functions, and whether functions can be taught to students in earlier grades (rather than starting at the high school level).

An Introduction to Hysteresis

Consider the dynamics of a thermostat, which is either in a state of being off or in a state of being on. Let -1 represent its off state and 1 represent its on state, and let u be the ambient temperature of a refrigerator. Suppose the thermostat is off if the temperature is less than 0 degrees Celsius and on if the temperature is greater than 0. So at 0 degrees Celsius, the thermostat switches from off to on. This behaviour is shown in figure 1a. On the other hand, suppose the thermostat switches from on to off when the temperature is 5 degrees as depicted in figure 1b. This means the path from off to on is different from on to off; that is, the dynamics of the thermostat is *path dependent*. This path dependence creates a loop as shown in figure 1c. This loop is known as a *hysteresis loop*, and we say the performance of the thermostat exhibits hysteresis.

In the case of the thermostat, the presence of hysteresis is a benefit. Consider figure 1a, which without the presence of hysteresis would mean any slight temperature change above or below zero causes the thermostat to frequently switch off and on. This would quickly wear down the components of the thermostat, and hence, the presence of hysteresis improves the performance and quality of a thermostat. Furthermore, in figure 1c, the state of the thermostat may be -1 or 1 for $u \in [0, 5]$; that is, for a particular value of u between 0 and 5, there are two possibilities, namely -1 and 1. This violates the definition of function noted above. Instead, in the presence of hysteresis, the state of the thermostat is determined by knowing previously whether it was off or on. This dependence on the past is known as the *memory effect* of hysteresis.

In the example of the thermostat, hysteresis appears in a human-made device. Hysteresis is more often observed in natural processes such as freezing-thawing, magnetism, population dynamics, potential energy, and ecosystem changes (Alimov, Kornev & Mukhamadullina, 1998; Aiki & Minchev 2005; Morris, 2011; Noori, 2014; Berdugo,Vidiella, Solé & Maestre, 2022). Since hysteresis is a phenomenon

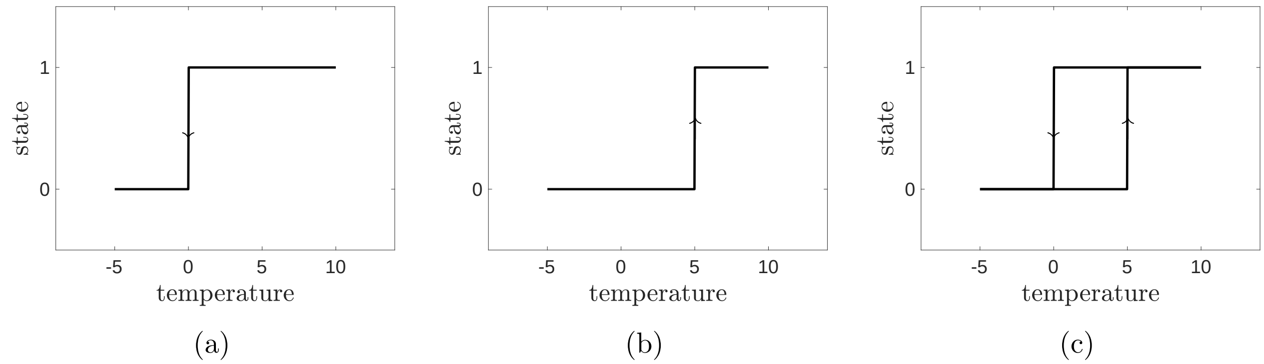

Figure 1: A thermostat with -1 as its off state and 1 as its on state, and u as temperature. (a) The thermostat switching from off to on when u=0. (b) The thermostat switching from on to off when u=5. (c) The dynamics shown in (a) and (b) plotted together.

that occurs in physical systems, it is commonly modelled by a differential equation,

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

where $x(t) \in \mathbb{R}$ is the solution to the differential equation, $t \in \mathbb{R}$ is time, and f is a continuous and differentiable mapping of x and u. In the context of determining whether a system exhibits hysteresis, $u(t) \in \mathbb{R}$ is the input of the system, and it affects the behaviour of $x(t)$, which is called the output. The relationship between the input and output is given by (1). The plane used to depict a hysteresis loop is an input-output graph as shown in Figure 2. The horizontal axis has been labelled 'input' and the vertical axis has been labelled 'output', but they can also be symbolized by u and x, respectively. The specific values of the input and output do not add to the understanding of the discussions presented, so numerical scales along the axes have been omitted.

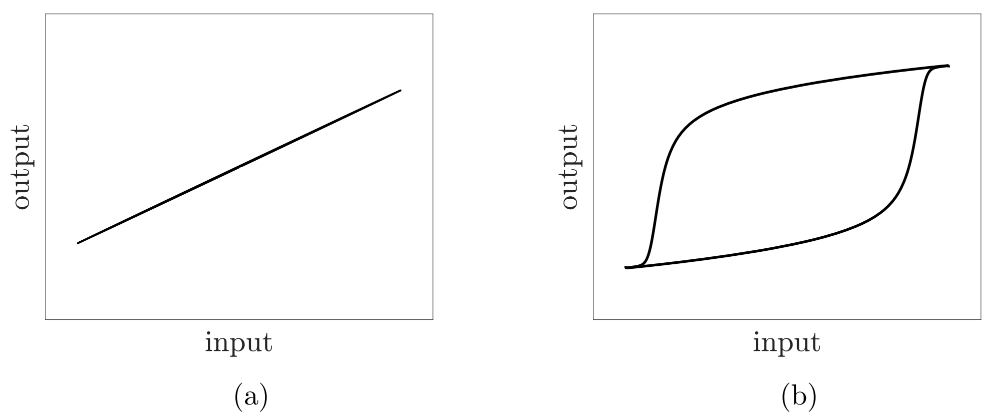

Figure 2: (a) Input-output graph displaying no looping behaviour. (b) Input-output graph displaying a hysteresis loop.

The thermostat example is discrete in that values of its output are either 1 or -1, and hence not a suitable solution to (1), which has continuous solutions for $x(t)$. For a continuous example, consider the process of water freezing and thawing in an object. The output is the flow rate of the water, and this depends on the temperature, which is the input. The input-output graph of these dynamics are shown in figure 3a if freezing is exactly the reverse of thawing. In other words, the frozen object as it is thawing has initial frozen state labelled by F, and this state (measured by the flow rate of the water, which is the output) changes as the input (ie. temperature) increases until reaching its final thawed state denoted by W, and reversing these dynamics, freezing is the same curve in figure 3a but from W to F if freezing is exactly the reverse of thawing.

If freezing is not exactly the reverse of thawing, the curve from F to W for thawing cannot be used to represent freezing from W to F. Figure 3b depicts the difference from F to W (in black) as compared with from W to F (in gray). These dynamics result in a hysteresis loop, and we say the freezing-thawing process exhibits hysteresis. In the examples of freezing and thawing, and the performance of the thermostat, there is a repetition between hot and cold temperatures that triggers each process to cycle over time. This repetitive nature is needed to test for hysteresis in a physical system, and consequently, the input is a periodic function (e.g. $u(t) = \sin(t)$ or $u(t) = 0.1\cos(2t)$). Additional details about input-output graphs for hysteresis loops can be found in Morris (2011) and Oh & Bernstein (2005).

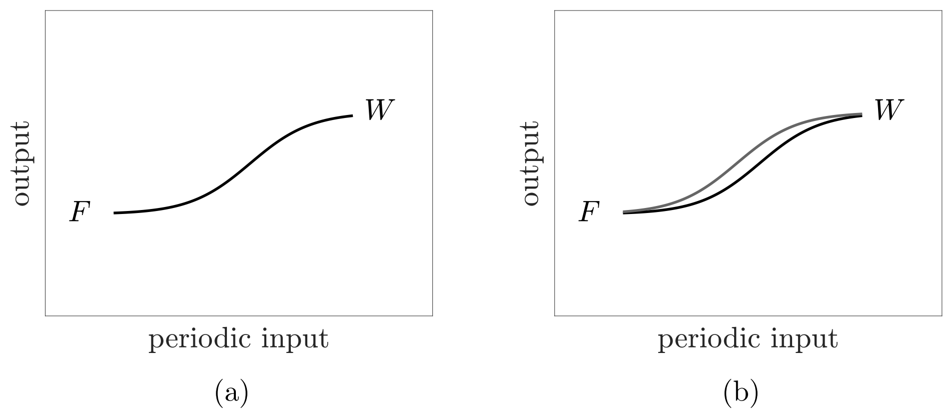

Figure 3: For the process of freezing and thawing, the depiction in (a) illustrates if freezing and thawing are processes that are exactly the reverse of each other, and (b) shows the case when they are not.

It has been shown that freezing is not exactly the reverse of thawing in Alimov, Kornev & Mukhamadullina (1998). This may be due to external factors such as different evaporation rates of water between freezing and thawing, and the fact water expands when it freezes. Alimov, Kornev & Mukhamadullina (1998) conducted physical experiments of temperature changes in pipes, and supported their experimental results with corresponding mathematical models. Their model describes changes in heat transfer and the Peclet number, which measures the rate of advection and diffusion of the water. Based on this work, they concluded "the linking conditions differ from the division conditions by a perceptible amount" (1998, p. 4). To clarify, the authors in Alimov, Kornev & Mukhamadullina (1998) refer to freezing as linking and thawing as division.

Despite the differences in shapes of hysteresis loops observed in figures 1c, 2b and 3b, the structure of all hysteresis loop has the appearance of one path *lagging* behind another. This observation led to the name hysteresis because the etymology of hysteresis means to lag behind another (Morris, 2011).

Path dependence, the memory effect and lagging offer several ways to characterize hysteresis, and they inspire the following colloquial definitions for hysteresis.

> Hysteresis is a process that follows a different path forward than backward when the process is reversed. This implies hysteresis is a process that exhibits path dependence.
>
> Hysteresis is a phenomenon that depends on its past behaviour to determine its current behaviour. In this context, we say hysteresis has a memory.
>
> Hysteresis describes a system whose output lags behind itself as the system input changes.

Additional definitions for hysteresis can be found in Brokate & Sprekels (1996), Morris (2011) and Oh & Bernstein (2005).

An Application for Non-Functions

A typical discussion of non-functions is usually either the graph of a circle or an arbitrary curve failing the vertical line test, which has limited connections to real-world applications, and the main focus is usually about what *is* a function. Hysteresis loops have one particular value of the input leading to two possible values of the output, which is a direct contradiction to the definition of a function as noted above. Verifying this, in part (a) of figure 2 where there is an absence of hysteresis, the curve is the graph of a function since it passes the vertical line test. On the other hand in part (b) of figure 2, the hysteresis loop shown fails the vertical line test and hence does not represent the graph of a function. While hysteresis is a real application that cannot be modelled by a function, it is important to point out that in the exploration of hysteresis, the concept of a function emerged. All the graphs in figures 1a, 1b and 3a can be described by a function. These graphs of functions lead to the construction of the hysteresis loops shown in figures 1c and 3b.

Inverse functions describe reversible processes (eg. adding five is precisely reversed by subtracting five), while hysteresis is an application that motivates functions without inverse functions. In the learning of inverse functions, there is again less of a focus on functions that do not have an inverse function, and applications of such. Hysteresis loops are visual representations of physical processes that are not exactly reversible as discussed in the examples of freezing-thawing and thermostat switching. In the case of the thermostat example, the presence of hysteresis (ie. its not exactly reversible nature) is a benefit. We may even extend this idea in hysteresis of graphing on the same plane a 'forward' process with its non-reversible 'backward' process, by graphing together a function and its inverse, and investigating whether common properties emerge, as it does for hysteresis in which looping always arises and hence leading to looping behaviour as a unifying characteristic for hysteresis.

An Application for the Geometry of Curves

Exploring the geometry of curves can be inspired by physical applications. Hysteresis loops can be an example for this. The previous discussions explained why closed curves (ie. loops) form in the input-output graphs of systems that exhibit hysteresis. Within this, there is quite a bit to investigate about the shape of hysteresis loops. For instance, self-crossing pinched hysteresis loops, as depicted in figure 4, are closed curves that are not simple. These type of hysteresis loops arise in engineering systems such as circuits and smart materials (Drinčić, Tan & Bernstein, 2011; Wang & Hui, 2017). The pinched hysteresis loop in figure 4 appears to exhibit some symmetry along its self-crossing; however, this may not always be the case and hence symmetry is another avenue for exploration (Wang & Hui, 2017). By considering the physical explanation of self-crossing in hysteresis loops, it becomes a motivation for why it is worthwhile to understand geometric concepts like symmetry and distinguishing simple curves from non-simple ones.

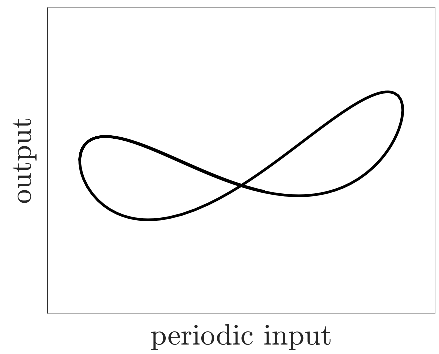

Figure 4: A pinched hysteresis loop, which is in the shape of a closed curve that is not simple.

Additional Suggestions for Applications

What has been presented in this article is a brief and modest discussion of hysteresis. There are many more facets to hysteresis worth considering, such as using hysteresis as a motivational example to examine the use of mathematics in computer programming. All the figures in this article are created from scripts written in MATLAB, and this requires mathematical knowledge of time scales, graphing and differential equations. Of course, using other software is possible and avenues for learning include writing code to generate hysteresis loops or phase planes of hysteretic systems.

Another application is using hysteresis loops to inspire learning about differentiability and the appearance of their corresponding graphs since some hysteresis loops are smooth (eg. figure 4) while some have cusps (eg. figure 2b). Other topics that can be investigated more deeply are defining hysteresis, modelling hysteresis in differential equations, historical perspectives of the first observations of hysteresis, and reflecting on the use of authentic practical applications as a tool for learning mathematics.

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