Hysteresis and Yarn Loops

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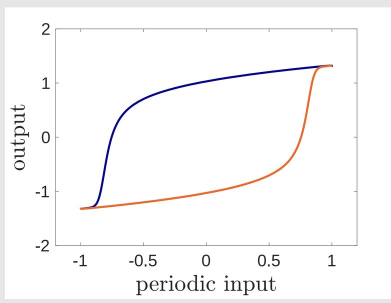
Highlights

- The shape of hysteresis loops provides fundamental properties of the dynamical system it comes from. The hysteresis loops in the examples below are generated in MATLAB.
- We also create hysteresis loops through the tactile medium of yarn, and highlight the hysteretic property of path dependence using different coloured yarn. The result is a unique tessellation pattern.
- Work done is a multidisciplinary project involving fundamental concepts from engineering, mathematics and the arts.

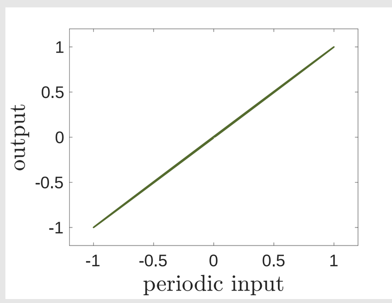
What is Hysteresis?

Hysteresis is a phenomenon occurring in dynamical systems that typically model natural and/or engineering applications [1, 2, 3, 4, 5]. One way to determine whether a dynamical system exhibits hysteresis is by introducing a periodic input into the system and then plotting the outputting behaviour of the system as the input varies.

If a simple closed curve appears and persists in this input-output map as the frequency of the periodic input approaches zero, the dynamical system is said to exhibit hysteresis [4]. Such a curve is called a hysteresis loop of the dynamical system. Since a hysteresis loop cannot be described by a function, systems that exhibit hysteresis are difficult to analyze. If the closed curve does not persist as the frequency of the periodic input goes to zero, then the system does not exhibit hysteresis. In this case, the resulting curve can be described by a function. See Figure 1.



(a) Hysteresis Present



(b) Hysteresis Absent

Figure 1: (a) A loop is formed when the output follows different paths as the input varies. This has been emphasized through the use of two different colours in the hysteresis loop. Consequently, hysteresis is synonymous with the idea of path dependence. (b) When a looping behaviour does not persist in the input-output map, this means the closed simple curve degenerates into a curve of a function, and indicates the system does not exhibit hysteresis.

Examples of Hysteresis Loops

Example 1: Consider a damped nonlinear second-order differential equation

$$\ddot{y}(t) + c\dot{y}(t) + k\left(y(t) - y^3(t)\right) = 0. \tag{1}$$

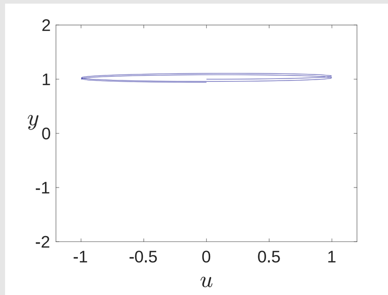
If the periodic input

$$u(t) = \sin(\omega t) \tag{2}$$

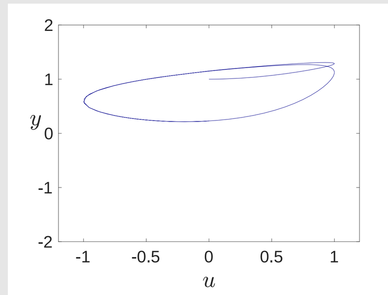
is introduced into (1), this leads to

$$\ddot{y}(t) + c\dot{y}(t) + k\left(y(t) - y^3(t)\right) = u(t). \tag{3}$$

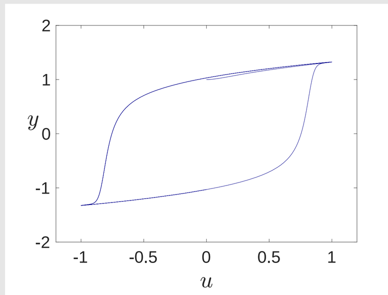
Figure 2 depicts the input-ouput curves of (1) with periodic input (2). From the figure, we see the appearance of simple closed curves and they persist even for small frequencies, ω, which indicates (1) exhibits hysteresis. Notice the shape of the hysteresis loop converges to a stable shape as ω approaches 0.



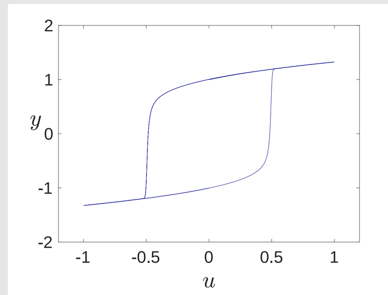
(a) $\omega = 1$



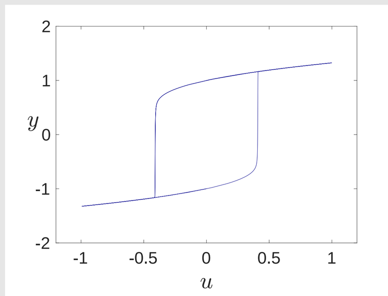
(b) $\omega = 0.1$



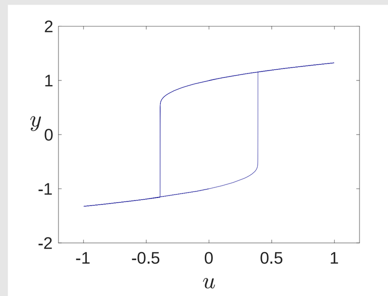
(c) $\omega = 0.01$



(d) $\omega = 0.001$



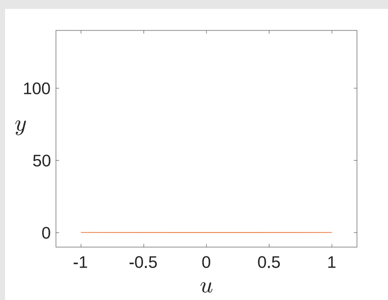
(e) $\omega = 0.0001$



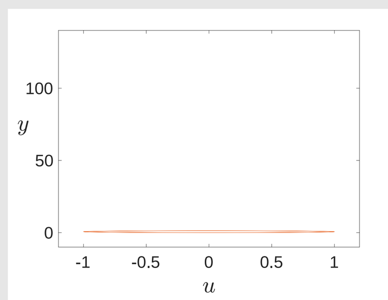
(f) $\omega = 0.00001$

Figure 2: Input-output curves of (1) with $c = 15$, $k = -1$ and initial condition $y(0) = 1, \dot{y}(0) = 0$ indicates hysteresis loops persist even for small ω.

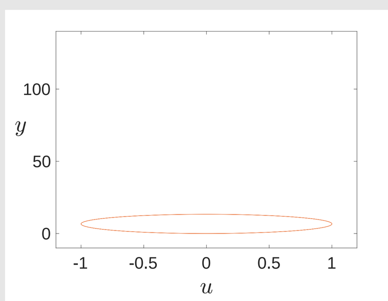
It turns out this system has three equilibria. This can be verified by rewriting (1) into first-order form and setting the derivatives to zero. The eigenvalues of (1) indicate two of the equilibria are stable and the third unstable. It is this scenario that causes the rapid transition (in other words, sharp jumps between the bottom and top of the hysteresis loop) displayed in Figure 2. In particular, as the input varies the system will tend to stay near a stable equilibrium (flat part of the hysteresis loop); however, if the input varies enough such that this causes the other stable equilibrium to be closer, the system will "jump" to the closer stable equilibrium. For a discussion of the relationship between hysteresis and stable equilibria, see [3].



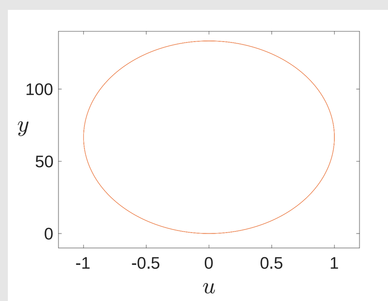
(a) $\omega = 1$



(b) $\omega = 0.1$



(c) $\omega = 0.01$



(d) $\omega = 0.001$

Figure 3: Input-output curves of (4) with $c = 15$ and initial condition $y(0) = 0, \dot{y}(0) = 0$ indicates hysteresis loops persist even for small ω.

Example 2: Suppose we let $k = 0$ in (1), which leads to the linear second-order differential equation

$$\ddot{y}(t) + c\dot{y}(t) = 0. \tag{4}$$

As in the previous example, Figure 3 depicts the input-output curves of (4) with periodic input (2). The figure shows that as ω goes to zero, simple closed curves appear and persists, indicating (4) is hysteretic. The appearance of jumps in the hysteresis loops (as in Example 1) does not happen in this case because this system has a continuum of stable equilibria, which causes the shape of the hysteresis loop to be smooth. For an example of a partial differential equation exhibiting smooth hysteresis loops, see [2].

Crocheting Hysteresis Loops

Due to the rotational symmetry in hysteresis loops, they can be used to form a unique tessellation pattern as depicted in Figure 4.

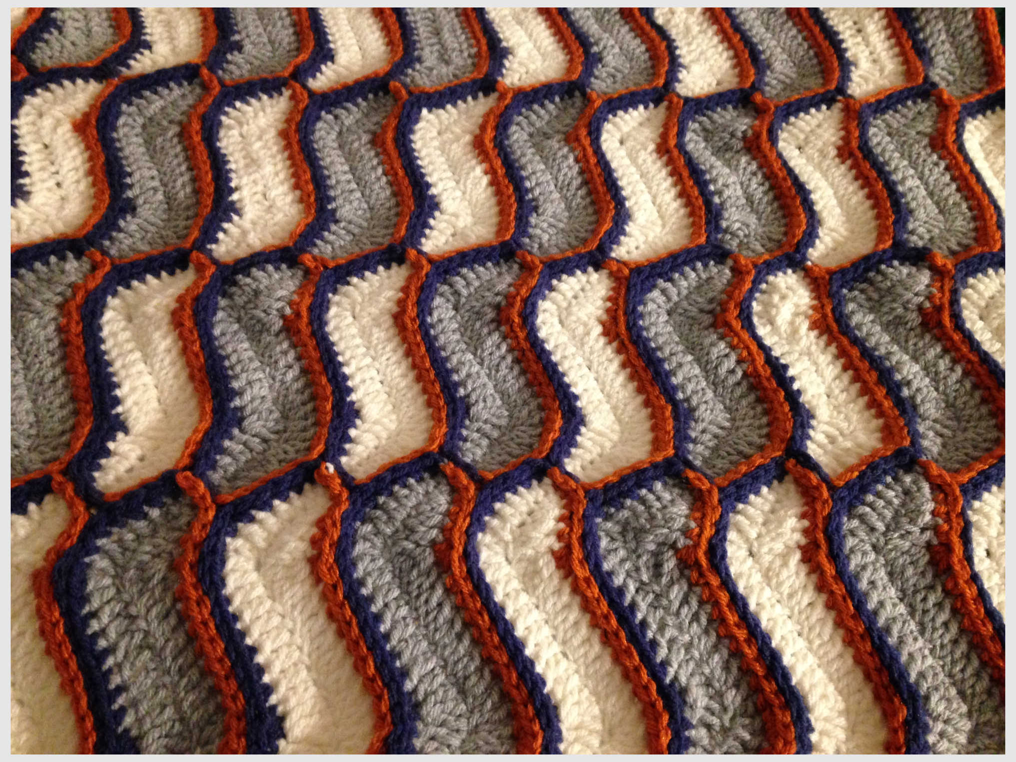


Figure 4: Tessellation pattern made of hysteresis loops.

The hysteresis loops are constructed from crocheting yarn, and this crochet pattern is new. Step-by-step images for crocheting a single hysteresis loop is noted in Figures 5 to 10. Creating hysteresis loops in this tactile way is a unique and artistic approach for exploring the shape of hysteresis loops.

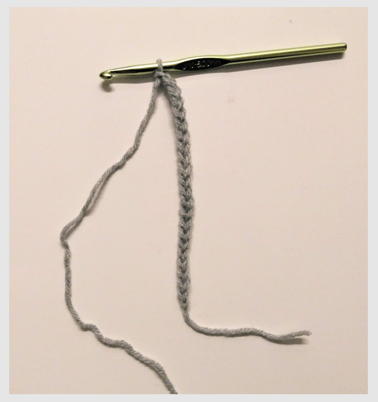


Figure 5: The foundation chain consists of 22 stitches using worsted weight (medium) yarn and a J/6mm or H/6.5mm sized hook.

Figure 6: The first row consists of single crochets, double crochets and half double crochets. This step constructs half of the interior of the hysteresis loop.

Figure 7: The second row is created on the other side of the foundation chain and consists of single crochets, double crochets and half double crochets. This step constructs the second half of the interior of the hysteresis loop with the foundation chain in the middle.

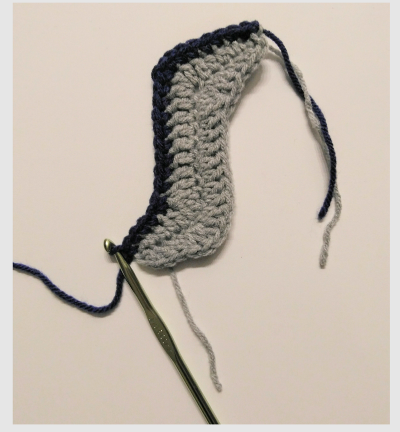


Figure 8: The first border consists of 23 single crochets along half the perimeter of the hysteresis loop.

Figure 9: The second border consists of 23 single crochets along the remaining half of the perimeter of the hysteresis loop. The two colours reinforce the notion of path dependence in hysteresis.

Figure 10: Cut yarn as needed and using a tapestry needle, weave in yarn ends to finish. The result is a crocheted hysteresis loop that is 15cm corner-to-corner, 13cm tall and 11cm wide. To form a tessellation pattern as in Figure 4, stitch the individual crocheted hysteresis loops together by fitting them next to one another so that there are no gaps and no overlaps.

Future Avenues

- Multiple definitions for hysteresis currently exist. Ideally, there should be one unifying rigorous definition for hysteresis.
- All the hysteresis loops presented here are closed simple curves. This need not be the case [5] and exploring non-simple closed curves could lead to more identifying properties of the dynamical system.
- Exploring further symmetries in the shape of hysteresis loops could lead to additional tessellation patterns.

References

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